MATHEMATICS

NOTES ON BANACH FUNCTION SPACES, VI

BY

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In the preceding five notes published in these Proceedings (Note I,66, p. 135–147; II, 66, p. 148–153; III, 66, p. 239–250; IV, 66, p. 251–263; V, 66, 496–504) we have discussed some of the fundamental properties of normed spaces L_ϱ of μ-measurable functions, and it was shown among other things that the real part $L_\varrho^{(r)}$ of L_ϱ is a Riesz space which is "complètement réticulé" if the seminorm ϱ has the weak Fatou null property. The present note is devoted to the study of abstract Riesz spaces. We first review briefly some of the main features of the theory, with special emphasis on some results of importance for the further investigation of the spaces L_ϱ.

16. *Riesz spaces*

Let L be a real linear vectorspace with elements $f, g, \ldots,$ (partially) ordered by $\leqslant$ such that

(i) $f \leqslant g$ implies $f+h \leqslant g+h$ for every $h \in L$,

(ii) $f \geqslant 0$ implies $af \geqslant 0$ for every real $a \geqslant 0$.

The subset $L^+ = \{f : f \in L,\ f \geqslant 0\}$ is said to be the *positive cone* of the *ordered linear vectorspace* L. Elements of L^+ are called positive elements and will usually be denoted by $u, v, w, \ldots$ in the following. If, for every pair $f, g \in L$, the least upper bound $\sup (f, g)$ and the greatest lower bound $\inf (f, g)$ with respect to the (partial) ordering exist in L, then L is said to be a *Riesz space*, or a *linear vector lattice*. The notion of a Riesz space is essentially due to F. RIESZ ([8], [9]). We list some examples.

Example 16.1. (i) L is the linear vectorspace of all real finite-valued functions $f(x)$ on the (non-empty) point set X, and $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in X$. Similarly, if μ is a countably additive measure in X, and we let L be the vectorspace of all real (almost everywhere finitevalued) μ-measurable functions on X, with identification of functions which are equal almost everywhere.

(ii) L is the vectorspace of all real continuous functions on the compact topological space X.

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(iii) Let ϱ be an arbitrary function seminorm, and L the real linear vectorspace $L_\varrho^{(r)}$ of all equivalence classes $[f] \in L_\varrho$ corresponding to real functions f. The ordering is defined by the convention that $[f] \leqslant [g]$ if and only if $f \prec g$ according to the notations of section 3 (Note I).

(iv) Let μ be a finitely additive signed measure on the field Γ of subsets of the (non-empty) point set X such that $\sup(|\mu(A)| : A \in \Gamma)$ is finite. Under the natural definition of addition and multiplication by real constants the collection L of all such μ is a real linear vectorspace, (partially) ordered if $\mu_1 \leqslant \mu_2$ denotes that $\mu_1(A) \leqslant \mu_2(A)$ for all $A \in \Gamma$. If $\mu_1, \mu_2 \in L$ it is easy to verify that $\nu = \sup(\mu_1, \mu_2)$ exists in L; indeed,

$$\nu(A) = \sup(\mu_1(B) + \mu_2(A - B) : A \supset B \in \Gamma).$$

Similarly, $\inf(\mu_1, \mu_2)$ exists in L.

It is customary to introduce the abbreviations $f^+ = \sup(f, 0)$, $f^- = \sup(-f, 0)$ and $|f| = \sup(f, -f)$. Furthermore, if $\inf(|f|, |g|) = 0$, it is said that f and g are disjoint, and this is denoted by $f \perp g$. For a number of simple properties of Riesz spaces we refer to N. Bourbaki ([2], Ch. II) or H. Nakano ([5], [6]).

The sequence $\{f_n : n \in N\}$ of elements of L is called increasing if $f_1 \leqslant f_2 \leqslant \ldots$, and decreasing if $f_1 \geqslant f_2 \geqslant \ldots$. This will be denoted by $f_n \uparrow$ or $f_n \downarrow$ respectively. If $f_n \uparrow$ and $f = \sup f_n$ exists in L, we write $f_n \uparrow f$. Similarly for a decreasing sequence. If $f_n \uparrow f$, then $f_{n_k} \uparrow f$ for any subsequence such that $n_k \uparrow \infty$. Furthermore, if $f_n \uparrow f$ and $g_n \uparrow g$, and a, b are positive constants, then $af_n + bg_n \uparrow af + bg$. If $u_n \in L^+$ for all $n \in N$ and $u_1 + \ldots + u_n \uparrow u$, we shall write $\sum_1^\infty u_n = u$. We recall that if $0 \leqslant u \leqslant \sum_1^\infty u_n$, then there exists $0 \leqslant u_n' \leqslant u_n$ (for every $n \in N$) such that $u = \sum u_n'$.

The Riesz space L is called *Dedekind complete* whenever every non-empty subset of L which is bounded from above has a least upper bound. Equivalently, L is Dedekind complete whenever every non-empty subset of L which is bounded from below has a greatest lower bound. Dedekind completeness of L is also indicated by saying that L is "complètement réticulé" (Bourbaki) or "universally continuous" (Nakano) respectively.

The indexed subset $\{f_\tau : \tau \in \{\tau\}\}$ of L is said to be directed upwards, or downwards, if for every pair $\tau_1, \tau_2 \in \{\tau\}$ there exists $\tau_3 \in \{\tau\}$ such that $f_{\tau_3} \geqslant \sup(f_{\tau_1}, f_{\tau_2})$, or $f_{\tau_3} \leqslant \inf(f_{\tau_1}, f_{\tau_2})$ respectively. If $\{f_\tau\}$ is directed upwards, we shall write $f_\tau \uparrow$. If $f_\tau \uparrow$ and $f = \sup f_\tau$ exists, we shall write $f_\tau \uparrow f$. Similarly, if $\{f_\tau\}$ is directed downwards.

Theorem 16.2. *The following statements are equivalent.*

(i) *L is Dedekind complete.*

(ii) *If $0 \leqslant u_\tau \uparrow$ and $u_\tau \leqslant v \in L$ for all τ, then there exists u such that $u_\tau \uparrow u$.*

(iii) *If $0 \leqslant u_\tau \downarrow$, then there exists u such that $u_\tau \downarrow u$.*

Of the spaces listed in Example 16.1, the spaces in (i) and (iv) are Dedekind complete, the space $L_\varrho^{(r)}$ in (iii) is Dedekind complete if ϱ has the weak Fatou null property (cf. Theorem 15.6 in Note V), and the space in (ii) is, in general, not Dedekind complete (if L is the space of all real continuous functions on the interval $[0, 1]$ with its usual topology, then L is not Dedekind complete).

17. *Linear subspaces*

Let K be a linear subspace of the Riesz space L. Then K is called a *Riesz subspace* if $f, g \in K$ implies that $\sup(f, g)$ and $\inf(f, g)$ are in K, K is called an *ideal* if $f \in K$ and $|g| \leqslant |f|$ implies that $g \in K$, and the ideal K is called a *normal subspace* if it follows from $f_\tau \uparrow f \in L$ and $f_\tau \in K$ for all τ that $f \in K$. Any normal subspace is an ideal (by definition), and it is easy to verify that any ideal is a Riesz subspace. In the Bourbaki terminology, a Riesz subspace is called "un sous-espace propre", an ideal is "un sous-espace épais", and a normal subspace is "une bande". In the Nakano terminology a Riesz subspace and an ideal are called a linear lattice manifold and a semi-normal subspace respectively, and Nakano's definition of a normal subspace coincides with the present one in the case that L is Dedekind complete.

It follows immediately from the definitions that an arbitrary set theoretic intersection of Riesz subspaces (or ideals, or normal subspaces) is a Riesz subspace (or an ideal, or a normal subspace). Hence, given the non-empty subset A of L, the smallest Riesz subspace (or ideal, or normal subspace) of L including A exists, and will be called the Riesz subspace (or ideal, or normal subspace) generated by A.

For the proofs of the following results we refer to N. BOURBAKI ([2], Ch. II, § 1.5) or to F. RIESZ ([8], [9]).

Theorem 17.1. *If $A \subset L$, then the set A^p of all $f \in L$ satisfying $f \perp g$ for all $g \in A$ is a normal subspace of L. In addition, $A \subset A^{pp}$, $A^p = A^{ppp}$ and $A^p \cap A^{pp} = \{0\}$.*

Theorem 17.2. *If L is Dedekind complete and K_1, K_2 are normal subspaces of L such that $K_1 \cap K_2 = \{0\}$, then the direct sum $K_1 \oplus K_2$ is a normal subspace of L. It follows that $L = K \oplus K^p$ for any normal subspace K, and for any $u \in L^+$ the component u_K in K is given by*

$$u_K = \sup(v : 0 \leqslant v \leqslant u,\ v \in K).$$

Corollary 17.3. *If L is Dedekind complete and $A \subset L$, then $L = A^p \oplus A^{pp}$, and A^{pp} is the normal subspace generated by A.*

Theorem 17.4. *Let L be Dedekind complete, $v \in L^+$ and K the normal subspace generated by the element v. For any $f \in L$ we set $f = f' + f''$ with $f' \in K$ and $f'' \in K^p$. Then, for any $u \in L^+$, we have*

$$\inf(u, nv) \uparrow u' \text{ as } n \to \infty.$$

Proof. Given $u \in L^+$, there exists $p \in L^+$ such that $\inf(u, nv) \uparrow p$. Obviously, $p \leqslant u$ and $p \in K$. On account of

$$\inf(u, nv) \geqslant \inf(p, nv) \geqslant \inf\{\inf(u, nv), nv\} = \inf(u, nv)$$

we have $\inf(u, nv) = \inf(p, nv)$ for all n, which implies that $\inf(p, nv) \uparrow p$. It follows then from

$$\inf(p, nv) + \inf(u-p, v) \leqslant \inf\{u, (n+1)v\} \leqslant p,$$

by taking the least upper bound on the left that $p + \inf(u-p, v) \leqslant p$, hence $\inf(u-p, v) = 0$, i.e., $u-p \in K^p$. This shows that $u = p + (u-p)$ is the desired decomposition of u, so $p = u'$, i.e., $\inf(u, nv) \uparrow u'$.

Corollary 17.5. *Let L be Dedekind complete, $v_n \in L^+$ for $n = 1, 2, \ldots$ with $v_n \uparrow$, and K the normal subspace generated by $\{v_1, v_2, \ldots\}$. For any $f \in L$ we set $f = f' + f''$ with $f' \in K$ and $f'' \in K^p$. Then, for any $u \in L^+$, we have*

$$\inf(u, nv_n) \uparrow u' \text{ as } n \to \infty.$$

Proof. Given $u \in L^+$, there exists $p \in L^+$ such that $\inf(u, nv_n) \uparrow p$. Obviously, $p \leqslant u$ and $p \in K$. As in the preceding proof we have $\inf(u, nv_n) = \inf(p, nv_n)$, so $\inf(p, nv_n) \uparrow p$. For $n \geqslant m$ (m fixed), it follows then from

$$\inf(p, nv_n) + \inf(u-p, v_m) \leqslant \inf(p, nv_n) + \inf(u-p, v_n) \leqslant$$
$$\leqslant \inf\{u, (n+1)v_{n+1}\} \leqslant p,$$

by taking the least upper bound (as $n \to \infty$) on the left that

$$p + \inf(u-p, v_m) \leqslant p,$$

hence $\inf(u-p, v_m) = 0$, i.e., $u - p \perp v_m$ for every m, so $u - p \in K^p$. This shows that $p = u'$, i.e., $\inf(u, nv_n) \uparrow u'$.

18. *Linear functionals*

The real linear functional φ on the Riesz space L is said to be *positive* whenever $\varphi(u) \geqslant 0$ for all $u \in L^+$. In this case we shall write $\varphi \geqslant 0$. Evidently, $\varphi \geqslant 0$ if and only if $f \leqslant g$ implies $\varphi(f) \leqslant \varphi(g)$.

Lemma 18.1. *Let τ be a function on L^+ into the real numbers such that $\tau(u) \geqslant 0$ and $\tau(u+v) = \tau(u) + \tau(v)$ for all $u, v \in L^+$. Then there exists a unique positive linear functional φ on L such that $\varphi(u) = \tau(u)$ on L^+.*

The real linear functional φ on L is said to be *order bounded* if, for every $u \in L^+$, the number $\sup(|\varphi(f)| : |f| \leqslant u)$ is finite. The set of all order bounded linear functionals will be denoted by $L^\sim$, and evidently $L^\sim$ is a real linear vectorspace under the natural definitions of addition and scalar multiplication. Note that every positive φ is in $L^\sim$. Indeed, if $u \in L^+$ is given and $|f| \leqslant u$, then $f^+ \leqslant u$ and $f^- \leqslant u$, so $|\varphi(f)| = |\varphi(f^+) - \varphi(f^-)| \leqslant \varphi(f^+) + \varphi(f^-) \leqslant 2\varphi(u)$, and hence $\sup(|\varphi(f)| : |f| \leqslant u) \leqslant 2\varphi(u) < \infty$.

Theorem 18.2 (*Jordan decomposition theorem*). *The linear functional φ on L is order bounded if and only if φ is the difference of two positive linear functionals.*

Proof. If $\varphi=\varphi_1-\varphi_2$ with φ_1 and φ_2 positive, then it is evident that $\varphi \in L^{\sim}$. For the proof of the converse we refer to N. Bourbaki ([2], Ch. II, § 2.2), where it is shown that if, for any $u \in L^+$, we define

$$\varphi^+(u)=\sup\,(\varphi(v)\,:\,0\leqslant v\leqslant u),$$

then $\varphi^+(u)\geqslant 0$ and $\varphi^+(u+u')=\varphi^+(u)+\varphi^+(u')$, so φ^+ can be extended to a positive linear functional on L which we denote again by φ^+. Since, obviously, $\varphi^+(u)\geqslant\varphi(u)$ on L^+, we have $\varphi=\varphi^+-\varphi^-$ with φ^+ and $\varphi^-=\varphi^+-\varphi$ positive.

The positive linear functionals φ^+ and φ^- in this proof are called the *positive variation* of φ and *negative variation* of φ respectively, and $|\varphi|=\varphi^++\varphi^-$ is said to be the *total variation* of φ. If $\varphi\geqslant 0$, then $\varphi^+=\varphi$, so $\varphi^-=0$ and $|\varphi|=\varphi^+=\varphi$. Also, if $\varphi_2-\varphi_1\geqslant 0$, then $\varphi_2{}^+-\varphi_1{}^+\geqslant 0$.

Theorem 18.3. *If φ is order bounded, then*

$$|\varphi|(u)=\sup\,(\varphi(f)\,:\,|f|\leqslant u)=\sup\,(|\varphi(f)|\,:\,|f|\leqslant u),$$

and hence $|\varphi(f)|\leqslant|\varphi|(|f|)$ on L.

By definition, we shall write $\varphi_1\leqslant\varphi_2$ for φ_1 and φ_2 order bounded whenever $\varphi_2-\varphi_1\geqslant 0$, and evidently $L^{\sim}$ is an ordered vectorspace with respect to the thus defined (partial) ordering.

Theorem 18.4. *$L^{\sim}$ is a Dedekind complete Riesz space with respect to the introduced (partial) ordering.*

Proof. We refer again to N. Bourbaki ([2], Ch. II, § 2.2), where it is shown firstly that $L^{\sim}$ is a Riesz space with $\sup\,(\varphi_1,\varphi_2)=(\varphi_2-\varphi_1)^++\varphi_1$, and secondly that if $\{\varphi_\alpha : \alpha\in\{\alpha\}\}$ is a non-empty set of positive elements of $L^{\sim}$ which is bounded from above and directed upwards, then $\sup\varphi_\alpha$ exists in $L^{\sim}$. Indeed, if $\tau(u)=\sup\varphi_\alpha(u)$ for every $u\in L^+$, then τ is shown to satisfy the hypotheses of Lemma 18.1, and hence the extension of τ onto all of L is the desired least upper bound.

Note that it follows from $\sup\,(\varphi_1,\varphi_2)=(\varphi_2-\varphi_1)^++\varphi_1$ that $\sup\,(0,\varphi)=\varphi^+$. Similarly, $\sup\,(-\varphi,0)=\varphi^+-\varphi=\varphi^-$. Hence, the already introduced notations φ^+ and φ^- are in agreement with the Riesz space notations.

The main results in this section are essentially due to F. Riesz ([8], [9]).

19. *Extension of positive linear functionals*

There do not always exist nonzero positive functionals on an arbitrary Riesz space L, and hence it may occur that $L^{\sim}$ consists only of the null functional. By way of example, if μ is an atomless countably additive finite measure in the point set X, and L is the vectorspace of all real

μ-measurable functions on X, then L is a Riesz space under the natural ordering, and it is well-known that any positive linear functional on L is identically zero. Our first aim in the present section is to derive a condition which is necessary and sufficient for the existence of a nonzero positive linear functional on L.

Let ϱ be a function on L^+ into the real numbers such that

(i) $0 \leqslant \varrho(u) < \infty$, $\varrho(u+v) \leqslant \varrho(u) + \varrho(v)$ and $\varrho(au) = a\varrho(u)$ for all $u, v \in L^+$ and all $a \geqslant 0$,

(ii) $0 \leqslant u \leqslant v$ implies $\varrho(u) \leqslant \varrho(v)$.

If, for every $f \in L$, we define $\varrho(f) = \varrho(|f|)$, then ϱ is evidently a seminorm on L satisfying the extra condition (ii). The function ϱ is called a *Riesz seminorm* on L.

If φ is a linear functional on L such that $|\varphi(f)| \leqslant \varrho(f)$ on L, then φ is order bounded. Indeed, for any $u \in L^+$ we have

$$\sup\,(|\varphi(f)| \,:\, |f| \leqslant u) \leqslant \sup\,(\varrho(|f|) \,:\, |f| \leqslant u) = \varrho(u) < \infty.$$

In addition, it follows by means of Theorem 18.3 that $|\varphi|(u) \leqslant \varrho(u)$, so

$$||\varphi|(f)| \leqslant |\varphi|(|f|) \leqslant \varrho(|f|) = \varrho(f).$$

Conversely, if φ is order bounded, then $\varrho(f) = |\varphi|(|f|)$ is a Riesz seminorm on L.

Theorem 19.1. *There exists a nonzero positive linear functional on L if and only if there exists a nonzero Riesz seminorm on L.*

Proof. If φ is a nonzero positive linear functional on L, then $\varrho(f) = \varphi(|f|)$ is a nonzero Riesz seminorm on L. Conversely, if ϱ is a nonzero Riesz seminorm on L, then $\varrho(u_0) > 0$ for some $u_0 \in L^+$, and $\psi(au_0) = a\varrho(u_0)$ is a linear functional on the real multiples of u_0, satisfying $|\psi(au_0)| \leqslant \varrho(au_0)$. By the Hahn–Banach extension theorem ψ can be extended on all of L such that $|\psi(f)| \leqslant \varrho(f)$ for all $f \in L$; hence, ψ is order bounded. Then $\varphi = \psi^+$ is positive and nonzero since $\varphi(u_0) = \psi^+(u_0) \geqslant \psi(u_0) = \varrho(u_0) > 0$.

We will prove a sharper result. If ϱ is a Riesz seminorm on L, and φ a positive linear functional defined on the Riesz subspace $K \subset L$ and satisfying $|\varphi(f)| \leqslant \varrho(f)$ on K, then φ may be extended on all of L such that the extended φ satisfies $|\varphi(f)| \leqslant \varrho(f)$ on L. The positivity of φ, however, may get lost in the process, i.e., it is not sure that after the extension the equality

$$\varphi^+(u) = \sup\,(\varphi(v) \,:\, 0 \leqslant v \leqslant u) = \varphi(u)$$

holds for all $u \in K^+$. If K is an ideal the equality holds on K^+, since in this case $0 \leqslant v \leqslant u \in K$ implies $v \in K$. It is, therefore, interesting to observe that also if K is a Riesz subspace (and not necessarily an ideal), there is among the extensions of the initial φ at least one which preserves positivity.

Theorem 19.2. *Let ϱ be a Riesz seminorm on L, and φ a positive linear functional on the Riesz subspace $K \subset L$ such that $|\varphi(f)| \leqslant \varrho(f)$ on K. Then there exists a positive linear functional ψ on L such that $\psi = \varphi$ on K and $|\psi(f)| \leqslant \psi(|f|) \leqslant \varrho(f)$ on L.*

Proof. Note first that $p(f) = \varrho(f^+)$ is a sublinear functional on L, since $p(f+g) = \varrho\{(f+g)^+\} \leqslant \varrho(f^+ + g^+) \leqslant \varrho(f^+) + \varrho(g^+) = p(f) + p(g)$ and $p(af) = \varrho\{(af)^+\} = \varrho(af^+) = ap(f)$ for $a \geqslant 0$. Secondly, φ is majorized by p on K since $\varphi(f) \leqslant \varphi(f^+) \leqslant \varrho(f^+) = p(f)$ for any $f \in K$. By the extension theorem, there exists a linear functional ψ on L satisfying $\psi = \varphi$ on K and $\psi(f) \leqslant p(f)$ on L. If $u \in L^+$, then $\psi(-u) \leqslant p(-u) = \varrho\{(-u)^+\} = \varrho(0) = 0$, so $\psi(u) \geqslant 0$.

In particular it follows that if ϱ is a Riesz seminorm such that $\varrho(v) > 0$ for a given $v \in L^+$, then there exists a positive linear functional φ on L such that $\varphi(v) = \varrho(v) > 0$ and $|\varphi(f)| \leqslant \varrho(f)$ on L. We shall prove a sharper result.

Lemma 19.3. *Let ϱ be a Riesz seminorm on L such that $\varrho(v) > 0$ for a given $v \in L^+$. Then*

$$\varrho_1(u) = \lim_{n\to\infty} \varrho\{\inf (u, nv)\}$$

is a Riesz seminorm on L such that $\varrho_1 \leqslant \varrho$, $\varrho_1(v) = \varrho(v) > 0$ and $\varrho_1(u) = 0$ for all $u \perp v$.

Proof. The proof is immediate if one observes (for the triangle inequality) that $\inf (u_1 + u_2, nv) \leqslant \inf (u_1, nv) + \inf (u_2, nv)$.

Corollary 19.4. *If ϱ is a Riesz seminorm on L such that $\varrho(v) > 0$ for a given $v \in L^+$, then there exists a positive linear functional φ on L such that $\varphi(v) = \varrho(v) > 0$, $\varphi(u) = 0$ for all $u \perp v$ and $|\varphi(f)| \leqslant \varrho(f)$ on L.*

The same method as applied to ϱ in Lemma 19.3 can be applied to a positive linear functional.

Lemma 19.5. *Given $v \in L^+$ and the positive linear functional φ on L, let $\varphi_v(u)$ be defined on L^+ by*

$$\varphi_v(u) = \lim_{n\to\infty} \varphi\{\inf (u, nv)\}.$$

Then $\varphi_v(u) \geqslant 0$ and $\varphi_v(u+u') = \varphi_v(u) + \varphi_v(u')$ for all $u, u' \in L^+$. Hence, φ_v can be extended on all of L as a positive linear functional such that $0 \leqslant \varphi_v \leqslant \varphi$. Note that $\varphi_v(u) = 0$ for all $u \perp v$.

Proof. The proof is similar to the proof of Lemma 19.3; observe in addition that $\varphi_v(u) + \varphi_v(u') \leqslant \varphi_v(u+u')$ follows from the inequality $\inf (u, nv) + \inf (u', mv) \leqslant \inf \{u + u', (m+n)v\}$.

It is possible to derive by means of this lemma some formules which are dual to the formulas $\varphi^+(u) = \sup (\varphi(v) : 0 \leqslant v \leqslant u)$ and $|\varphi|(u) = \sup (|\varphi(f)| : |f| \leqslant u)$. For Lemma 19.5 and the first dual formula cf. also I. Namioka ([7], Lemma 7.7 and Theorem 7.8).

Theorem 19.6. *If φ is a positive linear functional on L, and if $f \in L$, then*

$$\varphi(f^+) = \max\,(\psi(f) : 0 \leqslant \psi \leqslant \varphi),$$

$$\varphi(|f|) = \max\,(|\psi(f)| : |\psi| \leqslant \varphi).$$

Proof. For $0 \leqslant \psi \leqslant \varphi$ we have $\psi(f) \leqslant \psi(f^+) \leqslant \varphi(f^+)$, so

$$\sup\,(\psi(f) : 0 \leqslant \psi \leqslant \varphi) \leqslant \varphi(f^+).$$

For the converse inequality, let φ_{f^+} be defined as in the preceding lemma, i.e., $\varphi_{f^+}(u) = \lim \varphi\{\inf\,(u, nf^+)\}$ as $n \to \infty$ for every $u \in L^+$. Then $0 \leqslant \varphi_{f^+} \leqslant \varphi$, so φ_{f^+} is an admissable ψ, and

$$\varphi_{f^+}(f) = \varphi_{f^+}(f^+) - \varphi_{f^+}(f^-) = \varphi(f^+) - 0 = \varphi(f^+).$$

For $|\psi| \leqslant \varphi$ we have $|\psi(f)| \leqslant |\psi|(|f|) \leqslant \varphi(|f|)$, so $\sup\,(|\psi(f)| : |\psi| \leqslant \varphi) \leqslant \varphi(|f|)$. For the converse, let $\psi = \varphi_{f^+} - \varphi_{f^-}$. Then $|\psi| = \sup\,(\varphi_{f^+} - \varphi_{f^-}, \varphi_{f^-} - \varphi_{f^+}) \leqslant \varphi$, and $\psi(f) = \varphi_{f^+}(f) - \varphi_{f^-}(f) = \varphi(f^+) + \varphi(f^-) = \varphi(|f|)$.

20. *Integrals*

The order bounded linear functional φ on the Riesz space L is said to be an *integral* (a continuous linear functional in Nąkano's terminology) whenever it follows from $0 \leqslant u_n \downarrow 0$ that $\varphi(u_n) \to 0$ as $n \to \infty$. Evidently, the collection $L_c^\sim$ of all integrals is a linear subspace of $L^\sim$; we shall say that $L_c^\sim$ is the *associate space* of L. It is also evident that $\varphi \in L^\sim$ is an integral if and only if $0 \leqslant u_n \uparrow u$ implies $\varphi(u_n) \to \varphi(u)$.

Theorem 20.1. *The following statements are equivalent.*

(i) $\varphi \in L_c^\sim$,

(ii) $\varphi^+ \in L_c^\sim$ *and* $\varphi^- \in L_c^\sim$,

(iii) $|\varphi| \in L_c^\sim$.

Proof. (i) $\Rightarrow$ (ii). It will be sufficient to show that $u_n \downarrow 0$ implies $\varphi^+(u_n) \to 0$. To this end, let $0 \leqslant v \leqslant u_1$. Then $v - \inf\,(v, u_n) = \inf\,(v, u_1) - - \inf\,(v, u_n) \leqslant u_1 - u_n$ implies by the definition of φ^+ that $\varphi\{v - \inf(v, u_n)\} \leqslant \leqslant \varphi^+(u_1 - u_n)$, so

$$0 \leqslant \varphi^+(u_n) \leqslant \varphi\{\inf\,(v, u_n)\} + \varphi^+(u_1) - \varphi(v). \tag{1}$$

Since $u_n \downarrow 0$, we have $\inf\,(v, u_n) \downarrow \inf\,(v, 0) = 0$, so $\varphi\{\inf\,(v, u_n)\} \to 0$, and hence (1) shows that $0 \leqslant \lim \varphi^+(u_n) \leqslant \varphi^+(u_1) - \varphi(v)$ holds for all v satisfying $0 \leqslant v \leqslant u_1$. This implies

$$0 \leqslant \lim \varphi^+(u_n) \leqslant \varphi^+(u_1) - \sup\,(\varphi(v) : 0 \leqslant v \leqslant u_1) = 0.$$

(ii) $\Rightarrow$ (i). Follows from $\varphi = \varphi^+ - \varphi^-$.

(ii) $\Rightarrow$ (iii). Follows from $|\varphi| = \varphi^+ + \varphi^-$.

(iii) $\Rightarrow$ (ii). Follows from $0 \leqslant \varphi^+ \leqslant |\varphi|$ and $0 \leqslant \varphi^- \leqslant |\varphi|$.

Theorem 20.2. *$L_c^\sim$ is a normal subspace of $L^\sim$.*

Proof. It follows easily from the results in the preceding theorem that $L_c^\sim$ is an ideal. It remains to show that if $0 \leqslant \varphi_\tau \in L_c^\sim$ for all $\tau \in \{\tau\}$ and $\varphi_\tau \uparrow \varphi$ in $L^\sim$, then $\varphi \in L_c^\sim$. To this end, assume that $u_n \downarrow 0$ in L, and let $\varepsilon > 0$. Since $\varphi(u_1) = \sup \varphi_\tau(u_1)$, there exists $\tau_0 \in \{\tau\}$ such that $0 \leqslant (\varphi - \varphi_{\tau_0})(u_1) < \varepsilon$, and hence $0 \leqslant (\varphi - \varphi_{\tau_0})(u_n) < \varepsilon$ for all $n \in N$. On account of $\varphi_{\tau_0} \in L_c^\sim$ we have $\lim \varphi_{\tau_0}(u_n) = 0$ as $n \to \infty$, and it follows easily that $\lim \varphi(u_n) = 0$, so $\varphi \in L_c^\sim$.

The element $\varphi \in L^\sim$ will be called a *singular functional* if $\varphi \perp \psi$ for all $\psi \in L_c^\sim$, i.e., if $\inf(|\varphi|, |\psi|) = 0$ for every integral ψ.

Theorem 20.3. *The set $L_s^\sim$ of all singular functionals is a normal subspace of $L^\sim$. In addition, $L^\sim = L_c^\sim \oplus L_s^\sim$; in other words, every $\varphi \in L^\sim$ has a unique decomposition $\varphi = \varphi_c + \varphi_s$ with $\varphi_c \in L_c^\sim$ and $\varphi_s \in L_s^\sim$.*

Proof. Since $L^\sim$ is Dedekind complete (cf. Theorem 18.4) and since $L_c^\sim$ is a normal subspace, the desired results follow from Theorems 17.1 and 17.2.

This splitting up of $L^\sim$ as a direct sum of the subspaces of integrals and singular functionals respectively generalizes a well-known situation in integration theory (cf., e.g., H. Gordon and E. R. Lorch, [3]).

Note that, for any $\varphi \in L^\sim$, we have $(\varphi_c)_c = \varphi_c$, $(\varphi_s)_s = \varphi_s$ and $(\varphi_c)_s = (\varphi_s)_c = 0$. Furthermore, it is easy to verify that $\varphi \in L_s^\sim$ if and only if $\varphi^+ \in L_s^\sim$ and $\varphi^- \in L_s^\sim$ or, equivalently, if and only if $|\varphi| \in L_s^\sim$.

In the last theorem it was shown that any $\varphi \in L^\sim$ has a unique decomposition $\varphi = \varphi_c + \varphi_s$ with $\varphi_c \in L_c^\sim$ and $\varphi_s \in L_s^\sim$. If $\varphi \geqslant 0$, we have by Theorem 17.2 that

$$\varphi_c = \sup(\psi : 0 \leqslant \psi \leqslant \varphi, \ \psi \in L_c^\sim). \tag{2}$$

We shall prove now another formula for φ_c showing that the decomposition of a positive φ into an integral and a singular functional is similar to the decomposition of a finitely additive measure into a countably additive measure and a purely finitely additive measure (cf. M. A. Woodbury [10], K. Yosida and E. Hewitt [11], and H. Bauer [1], § 2, Lemma 3).

Theorem 20.4. *Let $0 \leqslant \varphi \in L^\sim$ and $0 \leqslant u \in L$. Then*

$$\varphi_c(u) = \inf(\lim \varphi(u_n) : 0 \leqslant u_n \uparrow u). \tag{3}$$

Proof. The expression on the right hand side of (3) is similar to the expression for the Lorentz seminorm in section 7 of Note III, and for that reason we shall denote this expression, temporarily, by $\varphi_L(u)$. Evidently, $0 \leqslant \varphi_L(u) \leqslant \varphi(u)$, and we show first that φ_L is additive on L^+. The proof that $\varphi_L(u+v) \leqslant \varphi_L(u) + \varphi_L(v)$ is trivial. For the converse, assume that $u, v \in L^+$ and $0 \leqslant w_n \uparrow u+v$ with $\lim \varphi(w_n) < \varphi_L(u+v) + \varepsilon$. Writing $u_n = \inf(w_n, u)$ and $v_n = w_n - u_n$, it follows easily that $0 \leqslant u_n \uparrow u$ and $0 \leqslant v_n \uparrow v$, so

$$\varphi_L(u) + \varphi_L(v) \leqslant \lim \varphi(u_n) + \lim \varphi(v_n) = \lim \varphi(w_n) < \varphi_L(u+v) + \varepsilon.$$

Hence φ_L is nonnegative and additive on L^+, so φ_L is extendable as a positive linear functional on all of L by Lemma 18.1.

The definition formula for $\varphi_L(u)$ can be written as

$$(4) \qquad \varphi_L(u) = \inf \left(\sum_1^\infty \varphi(v_n) : v_n \in L^+, \sum v_n = u\right),$$

and this enables us to prove easily that φ_L is countably additive on L^+, i.e. if $v_n \in L^+$ and $\sum v_n = u$, then $\sum \varphi_L(v_n) = \varphi_L(u)$. To this end, let $\varepsilon > 0$ be given. For every $n \in N$, there exists $\sum_k v_{nk} = v_n$ such that $\sum_k \varphi(v_{nk}) < \varphi_L(v_n) + \varepsilon/2^n$. Then $w_n = \sum_{j=1}^n \sum_{k=1}^n v_{jk}$ satisfies $0 \leqslant w_n \uparrow u$, and

$$\varphi(w_n) = \sum_{j=1}^n \sum_{k=1}^n \varphi(v_{jk}) < \sum_{j=1}^\infty \varphi_L(v_j) + \varepsilon$$

for all n. It follows that $\varphi_L(u) \leqslant \sum_{j=1}^\infty \varphi_L(v_j)$, and since the inverse inequality is evident, we obtain $\varphi_L(u) = \sum \varphi_L(v_n)$. In other words, if $0 \leqslant u_n \uparrow u$, then $\varphi_L(u_n) \uparrow \varphi_L(u)$, which implies that φ_L is an integral.

Having shown thus that φ_L is an integral satisfying $0 \leqslant \varphi_L \leqslant \varphi$, it follows from (2) that $0 \leqslant \varphi_L \leqslant \varphi_c \leqslant \varphi$. Since $\varphi_c \leqslant \varphi$, the definition of φ_L implies that $(\varphi_c)_L \leqslant \varphi_L$. Also, since φ_c is an integral, the same definition implies that $(\varphi_c)_L = \varphi_c$. Hence, $\varphi_c = (\varphi_c)_L \leqslant \varphi_L$. Combining this with $\varphi_L \leqslant \varphi_c$, we obtain $\varphi_L = \varphi_c$.

Note that the expression for $\varphi_L(u)$ in (4) is similar to the expression for the seminorm ϱ_c in section 6 of Note III. For a seminorm ϱ we have in general that $\varrho_L \leqslant \varrho_c$, but here, due to the linearity of φ, the two corresponding expressions yield the same result.

Definition 20.5. *The Riesz space L is said to have the Egoroff property if, given any $u \in L^+$ and the countably many sequences $u_{nk} \uparrow_k u$ for $n \in N$, there exists a sequence $0 \leqslant v_m \uparrow u$, and for every pair (m, n) an index $j(m, n)$ such that $v_m \leqslant u_{nj}$.*

For Riesz spaces having the Egoroff property the greatest lower bound in the last theorem is attained.

Theorem 20.6. *Assume that L has the Egoroff property, and let $0 \leqslant \varphi \in L^\sim$ and $0 \leqslant u \in L$. Then*

$$\varphi_c(u) = \min\left(\lim \varphi(u_n) : 0 \leqslant u_n \uparrow u\right).$$

Proof. For every $n \in N$ there is a sequence $u_{nk} \uparrow_k u$ such that $\varphi(u_{nk}) \leqslant \varphi_c(u) + n^{-1}$ for all k. By the Egoroff property there is a sequence $0 \leqslant v_m \uparrow u$ such that $v_m \leqslant u_{nj}$ for $j = j(m, n)$. Hence $\varphi(v_m) \leqslant \varphi(u_{nj}) \leqslant \varphi_c(u) + n^{-1}$ for all m, n. This shows that $\varphi(v_m) \leqslant \varphi_c(u)$ for all m, and hence $\lim \varphi(v_m) \leqslant \varphi_c(u)$. The converse inequality is evident.

As a side result, we immediately obtain the following statement, essentially due to S. Koshi [4].

Corollary 20.7. *Let L have the Egoroff property and $0 \leqslant \varphi \in L_s^\sim$. Then the subset A_φ of L consisting of all $f \in L$ satisfying $\varphi(|f|) = 0$ is an* *ideal with the property that for every $f \in L$ there exists a sequence $f_n \in A_\varphi$ such that $f_n^+ \uparrow f^+$ and $f_n^- \uparrow f^-$.*

Proof. Since $\varphi \geqslant 0$, it is evident that A_φ is an ideal. Since $\varphi \in L_s^\sim$ we have $\varphi_c = 0$; hence, given $u \in L^+$, there exists by the preceding theorem a sequence $0 \leqslant u_n \uparrow u$ with $\varphi(u_n) \uparrow \varphi_c(u) = 0$, and so $\varphi(u_n) = 0$ for all $n \in N$. It follows that for any given $f \in L$ there exist sequences $u_n \uparrow f^+$ and $v_n \uparrow f^-$ with $\varphi(u_n) = \varphi(v_n) = 0$ for all $n \in N$. Set $f_n = u_n - v_n$. Since $0 \leqslant \inf(u_n, v_n) \leqslant \inf(f^+, f^-) = 0$, we have $u_n \perp v_n$, and so $u_n = f_n^+$ and $v_n = f_n^-$. Hence $f_n^+ \uparrow f^+$, $f_n^- \uparrow f^-$ and $\varphi(f_n^+) = \varphi(f_n^-) = 0$, so $\varphi(|f_n|) = 0$, i.e., $f_n \in A_\varphi$ for all $n \in N$.

Example 20.8. Let X be an arbitrary non-empty point set and L the Riesz space of all real finitevalued functions on X. If $x_0 \in X$, then $\varphi(f) = f(x_0)$ is a positive linear functional on L, and hence $L^\sim$ does not consist only of the null functional. We will show that $L^\sim$ consists only of integrals. For this purpose, let $0 \leqslant \varphi \in L^\sim$ and $u_n \downarrow 0$ in L. Note first that $u_n \downarrow 0$ is equivalent to monotone pointwise convergence of the sequence $u_n(x)$ on X to zero from above. We have to prove that $\varphi(u_n) \downarrow 0$, and hence it is no restriction of the generality to assume that $u_1(x) > 0$ on X. Assume now that $\varphi(u_n) \geqslant \varepsilon > 0$ for all n, and let $v_n = (u_n - \varepsilon u_1/ /\{2\varphi(u_1)\})^+$. At each $x \in X$ we have $v_n(x) = 0$ for $n \geqslant n_x$, and so $w(x) = \sum_1^\infty v_n(x)$ converges on X. This implies that $\varphi(w) < \infty$; on the other hand $v_n \geqslant u_n - - \varepsilon u_1/\{2\varphi(u_1)\}$, so $\varphi(v_n) \geqslant \varphi(u_n) - \varepsilon/2 \geqslant \varepsilon/2$ for all n, and hence $\varphi(w) = \infty$. Contradiction.

Other examples will follow in Note VII.

21. *Riesz annihilators and the subspace L^a*

Let L be a Riesz space and $L^\sim$ the Riesz space of all order bounded linear functionals on L. By $L_c^\sim$ and $L_s^\sim$ we denote the subspaces of all integrals and all singular functionals respectively.

For any subset $A \subset L$, the *Riesz annihilator* A^0 is defined by

$$A^0 = \{\varphi : \varphi \in L^\sim,\ \varphi(f) = 0 \text{ for all } f \in A\},$$

and evidently A^0 is a linear subspace of $L^\sim$. For any subset $B \subset L^\sim$, the *inverse Riesz annihilator* ${}^0B$ is defined by

$${}^0B = \{f : f \in L,\ \varphi(f) = 0 \text{ for all } \varphi \in B\},$$

and evidently ${}^0B$ is a linear subspace of L.

Theorem 21.1. (i) *If A is an ideal in L, then A^0 is a normal subspace of $L^\sim$.*

(ii) *If B is an ideal in $L^\sim$, then ${}^0B$ is an ideal in L.*

Proof. (i) We prove first that $\varphi \in A^0$ implies $|\varphi| \in A^0$. To this end, it is sufficient to show that $|\varphi|(u) = 0$ for every positive u in A. Given

$u \geqslant 0$ in A, it follows from $|f| \leqslant u$ that $f \in A$, and so $\varphi(f)=0$, which implies

$$|\varphi|(u)=\sup\,(|\varphi(f)|\,:\,|f|\leqslant u)=0.$$

Now, let $\psi \in L^\sim$, $\varphi \in A^0$ and $|\psi| \leqslant |\varphi|$. Then $|\varphi| \in A^0$, and so $|\psi| \in A^0$ trivially. It follows then from $|\psi(u)| \leqslant |\psi|(u)=0$ that $\psi(u)=0$ for every $u \geqslant 0$ in A, so $\psi \in A^0$. This shows that A^0 is an ideal.

It remains to prove that if $0 \leqslant \varphi_\tau \in A^0$ for all $\tau \in \{\tau\}$ and $\varphi_\tau \uparrow \varphi$ in $L^\sim$, then $\varphi \in A^0$. Since $\varphi_\tau(u)=0$ for every τ and every $u \geqslant 0$ in A, and since $\varphi(u)=\sup \varphi_\tau(u)$ by Theorem 18.4, it follows that $\varphi(u)=0$ for every $u \geqslant 0$ in A, so $\varphi \in A^0$.

(ii) We prove first that $f \in {}^0B$ implies $|f| \in {}^0B$. To this end, it is sufficient to show that $\varphi(|f|)=0$ for all positive φ in B. Given $\varphi \geqslant 0$ in B, it follows from $|\psi| \leqslant \varphi$ that $\psi \in B$, and so $\psi(f)=0$, which implies

$$\varphi(|f|)=\sup\,(|\psi(f)|\,:\,|\psi|\leqslant \varphi)=0.$$

Note that Theorem 19.6 has been used. Now, let $g \in L$, $f \in {}^0B$ and $|g| \leqslant |f|$. Then $|f| \in {}^0B$, and so $|g| \in {}^0B$ trivially. It follows then from $|\varphi(g)| \leqslant \varphi(|g|)=0$ that $\varphi(g)=0$ for all $\varphi \geqslant 0$ in B, so $g \in {}^0B$. This shows that ${}^0B$ is an ideal.

In the next theorem we characterize the inverse Riesz annihilator of $L_s^\sim$ in terms of a continuity property.

Theorem 21.2. *Let L^a be the subset of L consisting of all $f \in L$ such that $|f| \geqslant u_1 \geqslant u_2 \ldots \downarrow 0$ implies $\varphi(u_n) \to 0$ for every $\varphi \in L^\sim$. Then $L^a={}^0(L_s^\sim)$, and hence L^a is an ideal in L.*

Proof. Assume first that $f \in L^a$, and let $0 \leqslant \varphi \in L^\sim$. For any sequence $0 \leqslant u_n \uparrow |f|$ we have $|f|-u_n \downarrow 0$, so $\varphi(u_n) \uparrow \varphi(|f|)$ since $f \in L^a$. It follows that

$$\varphi_c(|f|)=\inf\,(\lim \varphi(u_n)\,:\,0 \leqslant u_n \uparrow |f|)=\varphi(|f|),$$

and hence $\varphi_s(|f|)=0$. In particular, if $0 \leqslant \varphi \in L_s^\sim$, then $\varphi=\varphi_s$, so $\varphi(|f|)=0$. This shows that $|f| \in {}^0(L_s^\sim)$, and since ${}^0(L_s^\sim)$ is an ideal by the preceding theorem, it follows that $f \in {}^0(L_s^\sim)$. Hence, $L^a \subset {}^0(L_s^\sim)$.

Conversely, let $f \in {}^0(L_s^\sim)$, so $|f| \in {}^0(L_s^\sim)$. Then $\varphi(|f|)=0$ for all $\varphi \in L_s^\sim$, in particular for all $\varphi \geqslant 0$ in $L_s^\sim$. Now, let $|f| \geqslant u_n \downarrow 0$. Then $\varphi(|f|)=0$ for $0 \leqslant \varphi \in L_s^\sim$, so $\varphi(u_n)=0$ for all $n \in N$. Hence, if $0 \leqslant \varphi \in L^\sim$, $\varphi=\varphi_c+\varphi_s$, it follows from $\varphi_c(u_n) \downarrow 0$ (by the definition of φ_c) and $\varphi_s(u_n)=0$ that $\varphi(u_n) \downarrow 0$. But then $\varphi(u_n) \to 0$ for every $\varphi \in L^\sim$, so $f \in L^a$. Hence, ${}^0(L_s^\sim) \subset L^a$.

If L is Dedekind complete we can give still another characterization of L^a. Assume that $\pi \equiv \{K_n : n \in N\}$ is an increasing sequence of normal subspaces of L (i.e., $K_n \subset K_{n+1}$ for all n) such that the normal subspace generated by all the K_n together is L itself. We shall call any sequence π of this kind an *exhausting sequence of normal subspaces.* For any $f \in L$ we have $f=f_{\pi n}+f'_{\pi n}$ with $f_{\pi n} \in K_n$ and $f'_{\pi n} \in K_n{}^p$, and it is easy to verify that if $0 \leqslant f \in L$, then $0 \leqslant f_{\pi n} \uparrow f$, so $f \geqslant f'_{\pi n} \downarrow 0$. Hence, if $f \in L^a$, then $\varphi(f'_{\pi n}) \to 0$ for every $\varphi \in L^\sim$. It is not immediately evident, however, that the converse holds, i.e., that $\varphi(f'_{\pi n}) \to 0$ for every exhausting sequence and every $\varphi \in L^{\sim}$ implies $f \in L^a$. In order to have an efficient notation we shall, temporarily, denote by L^{α} the subset of all $f \in L$ such that $\varphi(f'_{\pi n}) \to 0$ as $n \to \infty$ for every exhausting sequence π and every $\varphi \in L^{\sim}$.

Theorem 21.3. *If L is Dedekind complete, then $L^{\alpha} = L^a$.*

Proof. We need only prove that $L^{\alpha} \subset L^a$. For this purpose, let $0 \leqslant f \in L^{\alpha}$ and $f \geqslant u_n \downarrow 0$. Without loss of generality we may assume that the normal subspace generated by f is L itself. Now, let $0 \leqslant \varphi \in L^{\sim}$; we have to show that $\varphi(u_n) \to 0$. If $\varphi(f) = 0$ this is evident; assume therefore that $\varphi(f) \neq 0$. Given $\varepsilon > 0$, we set $\delta = \varepsilon/\{2\varphi(f)\}$. Since $p_n = (\delta f - u_n)^+ \uparrow \delta f$, the normal subspace generated by all the p_n together is L itself. Hence, denoting by K_n the normal subspace generated by p_n, the sequence $\pi \equiv \{K_n\}$ is exhausting, so $\varphi(f'_{\pi n}) < \varepsilon/2$ for $n \geqslant n_0$. Evidently, the component of $\delta f - u_n$ in K_n is p_n, so the component of $u_n - \delta f$ is $-p_n$, and it follows that the component $(u_n)_{\pi n}$ of u_n is $(\delta f)_{\pi n} - p_n$, so $0 \leqslant (u_n)_{\pi n} \leqslant \delta f$. The component $(u_n)'_{\pi n}$ in $K_n{}^p$ satisfies $0 \leqslant (u_n)'_{\pi n} \leqslant f'_{\pi n}$. Hence, for $n \geqslant n_0$, we have $\varphi(u_n) \leqslant \varphi(\delta f) + \varphi(f'_{\pi n}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This shows that $\varphi(u_n) \downarrow 0$, so $f \in L^a$.

It follows immediately from the definition of L^a that if A is an ideal in L and the space $A^{\sim}$ of all order bounded linear functionals on A consists only of integrals (i.e., $A^{\sim} = A_c^{\sim}$), then $A \subset L^a$. Indeed, let $f \in A$, $|f| \geqslant u_n \downarrow 0$ and $0 \leqslant \varphi \in L^{\sim}$. The restriction of φ on A is a positive linear functional on A, and hence an integral by hypothesis, so $\varphi(u_n) \downarrow 0$, which shows that $f \in L^a$. A sharper result can be obtained if $L^{\sim}$ separates the elements of L, i.e., if ${}^0L^{\sim} = \{0\}$.

Theorem 21.4. *Let ${}^0L^{\sim} = \{0\}$, and let A be an ideal in L such that $A^{\sim}$ can be identified to $L_c^{\sim}$ in the sense that*

(i) *the restrictions on A of different elements of $L_c^{\sim}$ are different,*

(ii) *every $\varphi \in A^{\sim}$ has an extension Φ onto all of L such that $\Phi \in L_c^{\sim}$.*

Then $A \subset L^a$, $A^0 = (L^a)^0 = L_s^{\sim}$, and $A^p = (L^a)^p$. In the particular case that L is Dedekind complete, the last statement is equivalent to the statement that A and L^a generate the same normal subspace.

Proof. It is evident by the remark preceding the theorem that $A \subset L^a$, so $A^0 \supset (L^a)^0$. Furthermore, the hypotheses (i), (ii) imply that if $0 \leqslant \varphi \in L^{\sim}$ and $\varphi = 0$ on A, then $\varphi = \varphi_s$, hence $A^0 \subset L_s^{\sim}$. It follows already that $(L^a)^0 \subset A^0 \subset L_s^{\sim}$. On the other hand the equality $L^a = {}^0(L_s^{\sim})$ implies that $(L^a)^0 \supset L_s^{\sim}$. Hence $A^0 = (L^a)^0 = L_s^{\sim}$.

It follows from $A \subset L^a$ that $A^p \supset (L^a)^p$, so it remains to show that $A^p \subset (L^a)^p$. Assume the existence of $f_1 \geqslant 0$ in A^p such that f_1 is not in $(L^a)^p$. This implies that for some $u \geqslant 0$ in L^a we have $f = \inf(f_1, u) \neq 0$. Note that $f \in A^p$ and $f \in L^a$. Since ${}^0L^{\sim} = \{0\}$, there exists $0 \leqslant \varphi \in L^{\sim}$ such that $\varphi(f) > 0$, and hence Lemma 19.5 shows (in view of $f \in A^p$) that we may assume $\varphi = 0$ on A. But then, as already observed in the preceding paragraph, $\varphi = \varphi_s$ on L, and so $\varphi \in L_s^{\sim}$. Since $f \in L^a = {}^0(L_s^{\sim})$, it follows then that $\varphi(f) = 0$. Contradiction.

Note that if we drop the hypothesis ${}^0L^{\sim}=\{0\}$, then it remains true that $A \subset L^a$ and $A^0=(L^a)^0=L_s^{\sim}$, but $A^p=(L^a)^p$ need no longer hold as shown, e.g., by the example that L is the space of all real finite Lebesgue measurable functions on the real line. Then $L^{\sim}=\{0\}$, $L^a=L$, and A can be any ideal in L.

In the next note we shall investigate normed Riesz spaces; some of the results in the present note can then be brought into a sharper form.

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